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Bubbling 1/2 BPS solutions of minimal six-dimensional supergravity

James T. Liu and Diana Vaman

Michigan Center for Theoretical Physics, Randall Laboratory of Physics, The University of Michigan, Ann Arbor, MI 48109–1120

E-mail: {jimliu,dvaman}@umich.edu

Abstract

We continue our previous analysis (hep-th/0412043) of 1/2 BPS solutions to minimal 6d supergravity of bubbling form. We show that, by turning on an axion field in the T^2 torus reduction, the constraint $F \wedge F$, present in the case of an $S^1 \times S^1$ reduction, is relaxed. We prove that the four-dimensional reduction to a bosonic field theory, whose content is the metric, a gauge field, two scalars and a pseudo-scalar (the axion), is consistent. Moreover, these reductions when lifted to the six-dimensional minimal supergravity represent the sought-after family of 1/2 BPS bubbling solutions.

1 Introduction

In this note we complete the search for bubbling 1/2 BPS solutions of minimal six dimensional supergravity initiated in [1]. These configurations were in turn motivated by [2], where 1/2 BPS supergravity solutions corresponding to bubbling-type deformations of $\mathrm{AdS}_5 \times S^5$ geometry were shown to admit a dual description in terms of free fermions [3, 2].

In [1], inspired by [2], we considered solutions of minimal six dimensional supergravity, which had an $S^1 \times S^1$ isometry $ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{H(x)+G(x)}d\phi_1^2 + e^{H(x)-G(x)}d\phi_2^2$, and $-2H_{(3)} = F_{(2)}d\phi_1 + \widetilde{F}_{(2)}d\phi_2$. We found that this metric ansatz did not allow the existence a family of 1/2 BPS solutions, because of an additional constraint $F_{(2)} \wedge F_{(2)} = 0$. We showed that this constraint arises for all metric reductions of the type $S^n \times S^n$, with the exception of the case $n = 3$, which was the case in [2]. From a bosonic reduction perspective, the $S^1 \times S^1$ reduction was inconsistent; it had to be supplemented by hand by the above mentioned constraint. Otherwise said, a particular six dimensional equation of motion, the Einstein equation with components $\phi_1\phi_2$ could not be recovered from the Lagrangian of the effective four dimensional bosonic theory, and it corresponded precisely to the constraint $F_{(2)} \wedge F_{(2)} = 0$. Therefore, we conjectured that by turning on an axion field, this constraint could in principle be eliminated. Interestingly enough, the $\mathrm{AdS}_3 \times S^3$, the maximally symmetric plane wave, and the multi-center string arising from a D1-D5 configuration were shown to satisfy the constraint.

Here we demonstrate that, indeed, a generic T^2 torus reduction ansatz

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^{H(x)+G(x)}d\phi_1^2 + e^{H(x)-G(x)}\left(d\phi_2 + \chi(x)d\phi_1\right)^2 \\ -2H_{(3)} &= F_{(2)} \wedge d\phi_1 + \widetilde{F}_{(2)} \wedge \left(d\phi_2 + \chi d\phi_1\right) \end{aligned} \tag{1.1}$$

not only eliminates the need for the constraint, but at the same time leads to the construction of the family of 1/2 BPS solutions which correspond to bubbling-type deformations of the $\mathrm{AdS}_3 \times S^3$ geometry. In (1.1) we denoted by $\chi(x)$ the additional field to be kept in the Kaluza-Klein truncation, which retains in the four dimensional bosonic effective field theory besides the metric, a gauge field ($F_{(2)}$ and $\widetilde{F}_{(2)}$ are related by the self-duality of the 3-form field strength $H_{(3)}$), two scalars H, G, and a pseudo-scalar, the axion χ. In contrast to the conclusion drawn in [1], where the rectangular torus reduction was inconsistent, we prove that by retaining the axion, we have achieved a consistent *bosonic* truncation. In order to be able to lift the bosonic four dimensional solutions to a family of supersymmetric solutions which includes $\mathrm{AdS}_3 \times S^3$, the Killing spinors must be charged under the two $U(1)$ symmetries.

It is worth noting that, while the bubbling AdS_5 solutions of type IIB supergravity were constructed in terms of a harmonic function which had to obey certain boundary conditions

in order for the solution to be non-singular, here we find that the bubbling AdS_3 solution is written in terms of two independent functions, each obeying second order differential equations.

The paper is structured following a similar pattern to [1]: in Section 2 we show that the T^2 reduction ansatz yields a consistent *bosonic* reduction of minimal six dimensional supergravity and furthermore perform a reduction of the gravitino supersymmetry variation. In Section 3 we construct the Killing spinor associated with this T^2 reduction, and obtain the sought-after family of 1/2 BPS solutions corresponding to a bubbling AdS_3. We conclude with a discussion section. Finally, the appendices contain the analysis of the integrability of the supersymmetry variations and the full set of differential and algebraic identities obeyed by the spinor bilinears.

2 Bosonic reduction of minimal $D = 6$ supergravity on T^2

As in [1], we are concerned with the reduction of $D = 6$, $\mathcal{N} = (1, 0)$ supergravity to yield an effective theory in four dimensions. Unlike [1], however, which focused on the $S^1 \times S^1$ reduction, we now consider the full T^2 reduction, allowing in particular a mixing between the two U(1) isometries, related to the tilting of the torus.

Although it is straightforward to couple to a tensor multiplet (which would be necessary for more general D1-D5 systems), here we consider only the minimal $\mathcal{N} = (1, 0)$ supergravity, consisting of the gravity multiplet $(g_{MN},\ \psi_M,\ B^+_{MN})$, where B^+_{MN} denotes a two-form potential with self-dual field strength, $H_{(3)} = dB^+_{(2)}$ and ψ_M is a left-handed gravitino satisfying the projection $\Gamma^7\psi_M = -\psi_M$. Here we are following the notation introduced in [1].

The bosonic equations of motion for the supergravity multiplet are simply

$$R_{MN} = \tfrac{1}{4} H_{MPQ} H_N{}^{PQ}, \qquad H_{(3)} = *H_{(3)}, \qquad dH_{(3)} = 0. \tag{2.1}$$

Although this theory does not admit a covariant Lagrangian formulation, we may formally take

$$e^{-1}\mathcal{L} = R - \frac{1}{2\cdot 3!} H^2_{(3)}, \tag{2.2}$$

with the addition that the self-duality condition on $H_{(3)}$ must be imposed by hand after obtaining the equations of motion.

Following [2,1], we proceed with a nearly standard Kaluza-Klein reduction on T^2, given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu + e^{H(x)}\left(e^{G(x)}d\phi_1^2 + e^{-G(x)}(d\phi_2 + \chi(x)d\phi_1)^2\right), \\ -2H_{(3)} &= F_{(2)}e^{-\frac{1}{2}(H+G)} \wedge e^{\underline{4}} + \widetilde{F}_{(2)}e^{-\frac{1}{2}(H-G)} \wedge e^{\underline{5}}. \end{aligned} \tag{2.3}$$

In writing $H_{(3)}$, and in the subsequent expressions, we use the natural vielbein basis

$$e^{\underline{4}} = e^{\frac{1}{2}(H+G)} d\phi_1, \qquad e^{\underline{5}} = e^{\frac{1}{2}(H-G)}(d\phi_2 + \chi d\phi_1). \tag{2.4}$$

Although we have written the $H_{(3)}$ ansatz in (2.3) in terms of four-dimensional gauge fields $F_{(2)}$ and $\widetilde{F}_{(2)}$, these fields are not independent, but are related by the condition that $H_{(3)}$ is self-dual. Computing

$$-2 * H_{(3)} = *_4 \widetilde{F}_{(2)} e^{-\frac{1}{2}(H+G)} \wedge e^{\underline{4}} - *_4 F_{(2)} e^{-\frac{1}{2}(H-G)} \wedge e^{\underline{5}}, \tag{2.5}$$

we see that the self-duality condition $H_{(3)} = *H_{(3)}$ implies

$$\widetilde{F}_{(2)} = -e^{-G} *_4 F_{(2)}, \qquad F_{(2)} = e^{G} *_4 \widetilde{F}_{(2)}. \tag{2.6}$$

Therefore, the effective bosonic reduction will result in a four-dimensional system consisting of the metric $g_{\mu\nu}$, a gauge field $F_{(2)}$, the two scalars H, G, and a pseudo-scalar 'axion' χ.

At this stage, it is worth commenting on the structure of the reduction ansatz. Recall that a standard T^2 reduction of the minimal $\mathcal{N} = (1, 0)$ theory would result in $\mathcal{N} = 2$ supergravity coupled to two vector multiplets in four dimensions. The two vector multiplets contain two scalars and two pseudoscalars, together parameterizing two $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ cosets, one related to the complex structure of T^2 and the other to its Kähler modulus. In contrast, here we set both metric gauge fields as well as the axionic scalar from the Kähler modulus to zero. Thus only the complex structure $\mathrm{SL}(2, \mathbb{R})$ survives, given by the complex parameter $\tau = \chi + ie^G$. The remaining scalar e^H parameterizes the volume of T^2, but is otherwise missing its axionic partner $\widetilde{\chi}$ ordinarily arising from an addition to the $H_{(3)}$ reduction ansatz in (2.3) of the form $(1+*) d\widetilde{\chi} \wedge e^{\underline{4}} \wedge e^{\underline{5}}$. Nevertheless, although this reduction is incomplete from a supersymmetric point of view (as it results in a non-supersymmetric field content), we will see below that it is a consistent reduction of the bosonic sector. The addition of the complex structure axion χ is crucial for consistency.

Proceeding with the bosonic reduction, we note that, in addition to the self-duality condition on $H_{(3)}$, the equation of motion $dH_{(3)} = 0$ results in the form-field equations

$$d\widetilde{F}_{(2)} = 0, \qquad dF_{(2)} + \widetilde{F}_{(2)} \wedge d\chi = 0. \tag{2.7}$$

It is this result here that indicates that $\widetilde{F}_{(2)} = d\widetilde{A}_{(1)}$ has a natural representation in terms of a potential, while $F_{(2)}$ has a more complicated representation. The form-field provides a source to Einstein's equations. We compute

$$\begin{aligned}
(H^2_{(3)})_{\mu\nu} &= \tfrac{1}{2} e^{-(H+G)} (F^2)_{\mu\nu} + \tfrac{1}{2} e^{-(H-G)} (\widetilde{F}^2)_{\mu\nu}, \\
(H^2_{(3)})_{\underline{44}} &= \tfrac{1}{4} e^{-(H+G)} F^2, \qquad (H^2_{(3)})_{\underline{55}} = \tfrac{1}{4} e^{-(H-G)} \widetilde{F}^2, \\
(H^2_{(3)})_{\underline{45}} &= \tfrac{1}{4} e^{-H} F_{\mu\nu} \widetilde{F}^{\mu\nu}.
\end{aligned} \tag{2.8}$$

Note that

$$H_{(3)}^2 = \tfrac{3}{4}e^{-(H+G)}F^2 + \tfrac{3}{4}e^{-(H-G)}\widetilde{F}^2. \tag{2.9}$$

However, by using the self-duality condition (2.6), we see simply that $H_{(3)}^2 = 0$, which is a kinematical constraint from self-duality.

Turning to the Einstein equations, we first compute the spin connections

$$\begin{aligned}
&\omega^{\underline{45}} = \tfrac{1}{2}e^{-G}d\chi,\\
&\omega^{\underline{4m}}e_{\underline{m}\mu} = -\tfrac{1}{2}\partial_\mu(H+G)e^{\underline{4}} - \tfrac{1}{2}e^{-G}\partial_\mu\chi e^{\underline{5}},\\
&\omega^{\underline{5m}}e_{\underline{m}\mu} = -\tfrac{1}{2}\partial_\mu(H-G)e^{\underline{5}} - \tfrac{1}{2}e^{-G}\partial_\mu\chi e^{\underline{4}},
\end{aligned} \tag{2.10}$$

as they also prove useful in the reducing the supersymmetry variations, below. It is then a straightforward exercise to compute the Riemann tensor through $R = d\omega + \omega\wedge\omega$, and then the Ricci tensor. In frame components, we obtain

$$\begin{aligned}
R_{\mu\nu} &= \hat{R}_{\mu\nu} - \tfrac{1}{2}(\partial_\mu H\partial_\nu H + \partial_\mu G\partial_\nu G + \partial_\mu\chi\partial_\nu\chi e^{-2G}) - \nabla_\mu\nabla_\nu H,\\
R_{\underline{44}} &= -\tfrac{1}{2}\partial^\mu H\partial_\mu(H+G) - \tfrac{1}{2}\Box(H+G) - \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi e^{-2G},\\
R_{\underline{45}} &= -\tfrac{1}{2}e^{-G}\left(\Box\chi + \partial_\mu(H-2G)\partial^\mu\chi\right),\\
R_{\underline{55}} &= -\tfrac{1}{2}\partial^\mu H\partial_\mu(H-G) - \tfrac{1}{2}\Box(H-G) + \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi e^{-2G}.
\end{aligned} \tag{2.11}$$

Combining these expressions with the source (2.8), we obtain the four-dimensional equations of motion

$$\begin{aligned}
&R_{\mu\nu} = \tfrac{1}{2}(\partial_\mu H\partial_\nu H + \partial_\mu G\partial_\nu G + e^{-2G}\partial_\mu\chi\partial_\nu\chi) + \nabla_\mu\nabla_\nu H + \tfrac{1}{4}e^{-(H-G)}\Big((\widetilde{F}^2)_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}\widetilde{F}^2\Big),\\
&\nabla^\mu\Big(e^H\nabla_\mu H\Big) = 0, \qquad \nabla^\mu\Big(e^H\nabla_\mu G\Big) = -e^{H-2G}\partial_\mu\chi\partial^\mu\chi + \tfrac{1}{8}e^G\widetilde{F}^2,\\
&\nabla^\mu\Big(e^{H-2G}\nabla_\mu\chi\Big) = -\tfrac{1}{16}\epsilon_{\mu\nu\rho\sigma}\widetilde{F}^{\mu\nu}\widetilde{F}^{\rho\sigma}.
\end{aligned} \tag{2.12}$$

The scalar equations were separated by taking appropriate linear combinations of the $R_{\underline{44}}$ and $R_{\underline{55}}$ equations.

We now see that the equations of motion, (2.7) and (2.12), may be derived from an effective four-dimensional Lagrangian

$$e^{-1}\mathcal{L} \;=\; e^H\Big[R + \tfrac{1}{2}\partial H^2 - \tfrac{1}{2}\partial G^2 - \tfrac{1}{2}\partial\chi^2 e^{-2G} - \tfrac{1}{8}e^{-(H-G)}\widetilde{F}^2 + \tfrac{1}{16}\chi\epsilon_{\mu\nu\rho\sigma}\widetilde{F}^{\mu\nu}\widetilde{F}^{\rho\sigma}\Big]. \tag{2.13}$$

The inclusion of the axion extends the analysis of [1], and removes the $F_{(2)}\wedge F_{(2)} = 0$ constraint. It is of course precisely $F_{(2)}\wedge F_{(2)}$ that sources the axion, and this is the origin of the inconsistency if the axion were to be truncated by hand.

2.1 Supersymmetry variations

Having completed the reduction of the bosonic sector with the axion, we now proceed to reduce the gravitino variation

$$\delta\psi_M = [\nabla_M + \tfrac{1}{48} H_{NPQ}\Gamma^{NPQ}\Gamma_M]\varepsilon. \tag{2.14}$$

Much of the analysis of the fermionic sector parallels that of [1]. However, some care must be taken when working with an off-diagonal metric on T^2. Following an identical Dirac decomposition

$$\Gamma_\mu = \gamma_\mu \times \sigma_1, \qquad \Gamma_{\underline{4}} = 1 \times \sigma_2, \qquad \Gamma_{\underline{5}} = \gamma_5 \times \sigma_1, \tag{2.15}$$

as well as the projection conditions $\Gamma^7\varepsilon = -\varepsilon$ and $\Gamma^7\psi_M = -\psi_M$ on Weyl spinors, the six-dimensional gravitino variation becomes

$$\begin{aligned}
\delta\psi_\mu &= [\nabla_\mu - \tfrac{i}{4}e^{-G}\partial_\mu\chi\gamma_5 + \tfrac{i}{16}e^{-\frac{1}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu]\epsilon,\\
\delta\lambda_H &= [\gamma^\mu\partial_\mu H + 2ie^{-\frac{1}{2}H}(e^{-\frac{1}{2}G}(\partial_{\phi_1} - \chi\partial_{\phi_2}) - i\gamma_5 e^{\frac{1}{2}G}\partial_{\phi_2})]\epsilon,\\
\delta\lambda_G &= [\gamma^\mu\partial_\mu G + ie^{-G}\partial_\mu\chi\gamma_5\gamma^\mu - \tfrac{i}{4}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu}\\
&\qquad + 2ie^{-\frac{1}{2}H}(e^{-\frac{1}{2}G}(\partial_{\phi_1} - \chi\partial_{\phi_2}) + i\gamma_5 e^{\frac{1}{2}G}\partial_{\phi_2})]\epsilon,
\end{aligned} \tag{2.16}$$

where we have defined the linear combinations

$$\lambda_H = 2(i\psi_{\underline{4}} + \gamma_5\psi_{\underline{5}}), \qquad \lambda_G = 2(i\psi_{\underline{4}} - \gamma_5\psi_{\underline{5}}). \tag{2.17}$$

(Note that these spinors were defined as χ_H and χ_G in [1]; here we use λ_H and λ_G to avoid confusion with the axion.) The four-dimensional Dirac spinor ϵ was related to the left-handed six-dimensional spinor by $\varepsilon = \epsilon \times \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

As highlighted in [1], to obtain a bubbling ansatz, we must allow for $\mathrm{U}(1)\times\mathrm{U}(1)$ charged Killing spinors. Thus we write

$$\epsilon(x,\phi_1,\phi_2) = e^{-\frac{i}{2}(\eta\phi_1+\tilde\eta\phi_2)}\epsilon(x), \tag{2.18}$$

where the Kaluza-Klein momenta (or chargers) η and $\tilde\eta$ are quantized in integer units. This quantization is enforced by the periodicity of the two-torus, even in the tilted case. In the $\mathrm{SL}(2,\mathbb{Z})$ point of view, the spinor charges $(\eta,\tilde\eta)$ transform as a doublet. The result of using a charged spinor is that we may make a simple replacement

$$\partial_\phi \to -i\tfrac{\eta}{2}, \qquad \partial_{\tilde\phi} \to -i\tfrac{\tilde\eta}{2}, \tag{2.19}$$

in the supersymmetry variations (2.16).

We thus see that, compared to the $S^1\times S^1$ case, the effect of working with T^2 is to introduce an axion χ (corresponding to an off-diagonal metric component $g_{\phi_1\phi_2}$) in both (2.13) and (2.16). Furthermore, the originally independent U(1) charges η and $\tilde\eta$ now combine into an $\mathrm{SL}(2,\mathbb{Z})$ doublet.

3 The bubbling AdS$_3$ solution

We now complete the supersymmetry analysis in the presence of the axion χ. Following [2,1], we introduce the spinor bilinears

$$
\begin{aligned}
&f_1 = \bar{\epsilon}\gamma^5\epsilon, \qquad f_2 = i\bar{\epsilon}\epsilon, \\
&K^\mu = \bar{\epsilon}\gamma^\mu\epsilon, \qquad L^\mu = \bar{\epsilon}\gamma^\mu\gamma^5\epsilon, \\
&Y_{\mu\nu} = i\bar{\epsilon}\gamma_{\mu\nu}\gamma^5\epsilon,
\end{aligned} \tag{3.1}
$$

where the factors of i are chosen to make these tensor quantities real. Using the methods of [4–7, 2, 1], we proceed to examine the algebraic and differential identities satisfied by the above tensors. The useful algebraic identities are straightforward:

$$
L^2 = -K^2 = f_1^2 + f_2^2, \qquad K \cdot L = 0. \tag{3.2}
$$

In addition, the complete set of differential identities are provided in Appendix B.

We first fix the form of the scalar quantities f_1 and f_2. Combining the differential identities for $\nabla_\mu f_1$ and $\nabla_\mu f_2$ in (B.1) with the L_μ identities in (B.2) and (B.3), we obtain

$$
\begin{aligned}
\partial_\mu f_1 &= \tfrac{1}{4}e^{-\frac{1}{2}(H+G)} * F_{\mu\nu}K^\nu + \tfrac{1}{2}f_2 e^{-G}\partial_\mu\chi = \tfrac{1}{2}f_1\partial_\mu(H-G) + f_2 e^{-G}\partial_\mu\chi, \\
\partial_\mu f_2 &= -\tfrac{1}{4}e^{-\frac{1}{2}(H+G)} F_{\mu\nu}K^\nu - \tfrac{1}{2}f_1 e^{-G}\partial_\mu\chi = \tfrac{1}{2}f_2\partial_\mu(H+G).
\end{aligned} \tag{3.3}
$$

This gives two equations for f_1 and f_2

$$
\partial_\mu[e^{-\frac{1}{2}(H-G)}f_1] = [e^{-\frac{1}{2}(H+G)}f_2]\partial_\mu\chi, \qquad \partial_\mu[e^{-\frac{1}{2}(H+G)}f_2] = 0, \tag{3.4}
$$

which may be integrated to obtain

$$
f_1 = (b + a\chi)e^{\frac{1}{2}(H-G)}, \qquad f_2 = ae^{\frac{1}{2}(H+G)}. \tag{3.5}
$$

In addition, the constants a and b are related through the identity $(\eta - \chi\widetilde{\eta})f_2 = -\widetilde{\eta}e^G f_1$ of (B.2). In particular

$$
a\eta + b\widetilde{\eta} = 0. \tag{3.6}
$$

Comparing with the $S^1 \times S^1$ compactification [1], we see that at this point the only effect of the axion is to shift f_1 in (3.5).

Given f_1 and f_2, we may now fix the normalization of the vectors K_μ and L_μ. Using (3.2), we obtain

$$
L^2 = -K^2 = f_1^2 + f_2^2 = e^H(a^2e^G + (b + a\chi)^2e^{-G}). \tag{3.7}
$$

Furthermore, the L_μ equations of (B.2) provide the constraints

$$
\eta L_\mu = b\partial_\mu e^H, \qquad \widetilde{\eta}L_\mu = -a\partial_\mu e^H, \tag{3.8}
$$

which are axion independent.

Following [2], we now observe from (B.1) that both $K_{(\mu;\nu)} = 0$ so that K^μ is a Killing vector and $dL = 0$. We thus choose a preferred coordinate basis so that the Killing vector $K^\mu\partial_\mu$ corresponds to $\partial/\partial t$ and the closed one-form $L_\mu dx^\mu$ to dy, where t and y are two of the four coordinates. In particular, we write down the four-dimensional metric as

$$ds_4^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + h_{ij}dx^i dx^j), \tag{3.9}$$

where $i, j = 1, 2$. The remaining components of the metric are V_i and h_{ij}, to be determined below, and h^2, given from (3.7) to be

$$h^{-2} = e^H(a^2 e^G + (b + a\chi)^2 e^{-G}). \tag{3.10}$$

In addition, for $L = dy$, (3.8) yields the constraints

$$\eta = b\partial_y e^H, \qquad \widetilde{\eta} = -a\partial_y e^H. \tag{3.11}$$

where we still allow for any of these constants η, $\widetilde{\eta}$, a or b to be zero.

Assuming $e^H = y$, which relates $\eta, \tilde{\eta}$ to a, b according to (3.11), from the supersymmetry variation of the gravitino $\delta\lambda_H$, we find

$$\left(\gamma^{\underline{3}}(\widetilde{\eta}^2 + (\eta - \chi\widetilde{\eta})^2)^{\frac{1}{2}} - (\eta - \chi\widetilde{\eta})e^{-G} + i\gamma_5\widetilde{\eta}\right)\epsilon = 0. \tag{3.12}$$

The projector (3.12) is easily solved by

$$\epsilon = \exp(i\alpha\gamma_5\gamma^{\underline{3}})\epsilon_1, \qquad \text{where} \quad \sinh(2\alpha) = \frac{\tilde{\eta}e^G}{\eta - \chi\tilde{\eta}}, \qquad \text{and} \quad \gamma^{\underline{3}}\epsilon_1 = \epsilon_1. \tag{3.13}$$

The norm of the spinor ϵ_1 is obtained from knowledge of the spinor bilinears f_1, f_2. Choosing a particular representation of the 4-dimensional Dirac matrices, let's say the chiral representation, we compute

$$\epsilon_1^T = (\epsilon_0, 0, -i\epsilon_0, 0), \qquad \text{with} \quad |\epsilon_0|^2 = \tfrac{1}{2}e^{\frac{1}{2}(H+G)}\sinh(2\alpha)^{-1} \tag{3.14}$$

and we can set the phase of ϵ_0 to zero (i.e. take ϵ_0 real).

There is another set of spinor bilinears which provides useful information, namely

$$\omega = \epsilon^T C\gamma_\mu\epsilon\; dx^\mu, \tag{3.15}$$

where C is the charge conjugation matrix $\gamma_\mu^T = -C\gamma^\mu C^{-1}$. One can check that the one-form ω is closed. Substituting the Killing spinor (3.13) into ω we obtain

$$\omega_{\underline{2}} = \frac{\cosh(2\alpha)e^{\frac{1}{2}(H+G)}}{\sinh(2\alpha)} = \frac{1}{\widetilde{\eta}}h^{-1}, \qquad \omega_{\underline{1}} = \frac{1}{\widetilde{\eta}}ih^{-1}, \tag{3.16}$$

where we used that for chiral representation $C = i\gamma^2\gamma^0$. Given that $\omega = h(\omega_{\underline{2}} e^{\underline{2}} + \omega_{\underline{1}} e^{\underline{1}})$ is closed we conclude yet again that the 2-dimensional space parameterized by x^1, x^2 is (conformally) flat.

We now have sufficient information to fix the form of the field strengths $F_{(2)}$ as well as dV. For $F_{(2)}$, we use the component relations

$$F_{\mu\nu}K^\nu = -2a\partial_\mu e^{H+G} - 2(b+a\chi)e^{H-G}\partial_\mu\chi, \qquad \widetilde{F}_{\mu\nu}K^\nu = -2(b+a\chi)\partial_\mu e^{H-G} - 2ae^{H-G}\partial_\mu\chi, \tag{3.17}$$

obtained from (3.3) as well as the explicit form of the metric (3.9) to find

$$\begin{aligned} F_{(2)} &= -2\Big[a\, de^{H+G} + (b+a\chi)e^{H-G}d\chi\Big] \wedge (dt+V) - 2h^2 e^G *_3 d\Big[(b+a\chi)e^{H-G}\Big], \\ \widetilde{F}_{(2)} &= -2d\Big[(b+a\chi)e^{H-G}\Big] \wedge (dt+V) + 2h^2 e^{-G} *_3 \Big[a\, de^{H+G} + (b+a\chi)e^{H-G}d\chi\Big], \end{aligned} \tag{3.18}$$

where $*_3$ denotes the Hodge dual with respect to the flat spatial metric. For dV, we take the antisymmetric part of $\nabla_\mu K_\nu$ in (B.1), written in form notation as

$$dK = \tfrac{1}{2}e^{-\frac{1}{2}(H+G)}(f_2 F_{(2)} - f_1 * F_{(2)}), \tag{3.19}$$

and substitute in the expressions for the Killing vector $K = -h^{-2}(dt+V)$ as well as for $F_{(2)}$. This gives both the known expression for h^{-2}, namely (3.10), as well as the relation

$$dV = -h^4 e^H *_3 \Big[2a(b+a\chi)dG + \Big((b+a\chi)^2 e^{-2G} - a^2\Big)d\chi\Big]. \tag{3.20}$$

Note that when $\chi = 0$ this reduces to $dV = -2abh^4e^H *_3 dG$, obtained in [2, 1].

Combining the expression for dV in (3.20) with that of h^2 in (3.10), we may re-express the one-forms dG and $d\chi$ in terms of dV and $d(h^2)$. This allows us to rewrite $F_{(2)}$ in a more suggestive manner

$$\begin{aligned} F_{(2)} &= -2\Big[a\, de^{H+G} + (b+a\chi)e^{H-G}d\chi\Big] \wedge (dt+V) - 2ae^{H+G}dV + 2(b+a\chi) *_3 e^H d(h^2), \\ \widetilde{F}_{(2)} &= -2d\Big[(b+a\chi)e^{H-G}\Big] \wedge (dt+V) - 2(b+a\chi)e^{H-G}dV - 2a *_3 e^H d(h^2). \end{aligned} \tag{3.21}$$

In addition, so long as $a \neq 0$, the expression for dV may be written as

$$dV = *_3 a^{-1}e^{-H} d\left(\frac{b+a\chi}{a^2e^{2G} + (b+a\chi)^2}\right), \qquad (a \neq 0), \tag{3.22}$$

where we have again used the form of h^2 in (3.10). For $a = 0$, on the other hand, we would instead find simply

$$dV = -b^{-2}e^{-H} *_3 d\chi, \qquad (a = 0). \tag{3.23}$$

The above results have all been obtained as a consequence of the Killing spinor equations. However, as is well known, for partial supersymmetry, the first order Killing spinor equations

generally imply only a subset of the complete equations of motion. This is indeed the case for the reduced $\mathcal{N}=(1,0)$ system. The relation between the Killing spinor equations and the bosonic equations of motion is investigated in Appendix A. The result of that analysis indicates that, so long as the $\widetilde{F}_{(2)}$ Bianchi identity $d\widetilde{F}_{(2)}=0$ and equation of motion $dF+\widetilde{F}\wedge d\chi=0$ are satisfied, we are then ensured a complete solution to the equations of motion.

Taking an exterior derivative of the expressions in (3.21), we obtain

$$\begin{aligned} d\widetilde{F}_{(2)} &= -2a\,d\Big[*_3 e^{H}d(h^2)\Big]-2(b+a\chi)e^{H-G}d^2V,\\ dF_{(2)}+\widetilde{F}_{(2)}\wedge d\chi &= -2ae^{H+G}d^2V+2(b+a\chi)\,d\Big[*_3 e^{H}d(h^2)\Big]. \end{aligned} \tag{3.24}$$

Note that dV is not automatically closed; this must be imposed as an additional consistency condition on either (3.22) or (3.23). We thus see that the bubbling AdS_3 analysis leads to two independent second order equations

$$d\Big[*_3 e^{H}d(h^2)\Big]=0,\qquad d\Big[*_3 e^{-H}dz\Big]=0, \tag{3.25}$$

where we have defined

$$z=\frac{1}{2ab}-\frac{b+a\chi}{a(a^2e^{2G}+(b+a\chi)^2)}=\frac{1}{2ab}\left(\frac{a^2(\chi^2+e^{2G})-b^2}{a^2e^{2G}+(b+a\chi)^2}\right), \tag{3.26}$$

so that

$$dV=-*_3\,e^{-H}dz. \tag{3.27}$$

Note that this expression remains valid in the limits $a\to 0$ or $b\to 0$, provided an (unimportant) infinite constant is subtracted.

The fact that there are now two second order equations, (3.25), indicates that the bubbling $\mathrm{AdS}_3\times S^3$ geometries have a different characteristic from that of the bubbling $\mathrm{AdS}_5\times S^5$ solutions of [2].

3.1 Specialization of η and $\widetilde{\eta}$

So far, we have left η and $\widetilde{\eta}$ unspecified and performed a general supersymmetry analysis. We now specialize the Killing spinor U(1) charges, considering the four possibilities for either of η and $\widetilde{\eta}$ vanishing or non-vanishing.

3.1.1 Both η and $\widetilde{\eta}$ non-vanishing

We begin with the case of both η and $\widetilde{\eta}$ non-vanishing. To be specific, we take $a=-\widetilde{\eta}=1$ as well as $b=\eta=1$, which was chosen to satisfy (3.6). In this case, (3.11) yields the simple result $e^H=y$, so that (3.10) becomes

$$h^{-2}=y(e^G+(1+\chi)e^{-G}). \tag{3.28}$$

We now see that, in the absence of the axion ($\chi = 0$), this expression reduces to that of [2,1], namely $h^{-2} = 2y\cosh G$. With $e^H = y$, the second order equations (3.25) reduce to

$$d\Big[*_3 y d(h^2)\Big] = 0, \qquad d\Big[*_3 y^{-1} dz\Big] = 0. \tag{3.29}$$

The first equation is a new one compared with the $\mathrm{AdS}_5 \times S^5$ case, and indicates that h^2 is harmonic in a four-dimensional auxiliary space $\mathbb{R}^2 \times \mathbb{R}^2$, restricted to s-waves only in the second $\mathbb{R}^2$. The second equation, on the other hand, is a direct generalization of the expression for z introduced in [2]. Thus z/y^2 is harmonic in a six-dimensional auxiliary space $\mathbb{R}^2 \times \mathbb{R}^4$, restricted to s-waves in the $\mathbb{R}^4$. In contrast with [2], however, the relation between z and G is now given by (3.26), and reads

$$z = \frac{1}{2}\frac{e^{2G} - (1-\chi^2)}{e^{2G} + (1+\chi)^2}, \tag{3.30}$$

which generalizes the expression $z = \frac{1}{2}\tanh G$ for a non-vanishing axion.

Note that the introduction of the axion has removed the $dG \wedge *_3 dG = dH \wedge *_3 dH$ that was identified in [1]. This, however, comes at the expense of introducing a second harmonic function to the bubbling AdS_3 construction.

To summarize, the bubbling $\mathrm{AdS}_3 \times S^3$ solution is given as:

$$\begin{aligned} ds_6^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \delta_{ij}dx^i dx^j) + y\Big[e^G d\phi_1^2 + e^{-G}(d\phi_2 + \chi d\phi_1)^2\Big], \\ \widetilde{F}_{(2)} &= -2\left[d\left((1+\chi)ye^{-G}\right) \wedge (dt+V) - h^2 e^{-G} *_3 \left(d(ye^G) + (1+\chi)ye^{-G}d\chi\right)\right], \end{aligned} \tag{3.31}$$

where

$$h^{-2} = y(e^G + (1+\chi)e^{-G}), \qquad z = \frac{1}{2}\frac{e^{2G} - (1-\chi^2)}{e^{2G} + (1+\chi)^2}, \qquad dV = -\tfrac{1}{y} *_3 dz. \tag{3.32}$$

The functions h^2 and z must satisfy the harmonic equations (3.29).

3.1.2 Only η non-vanishing

With the introduction of the axion, the spinor U(1) charges η and $\widetilde{\eta}$ are no longer interchangeable. Here we consider $\eta = 1$ and $\widetilde{\eta} = 0$. In this case, the constraint (3.6) indicates that $a = 0$. Avoiding the degenerate situation, we now take $b = \eta = 1$, so that (3.11) again gives $e^H = y$. This time, however, the relation (3.10) yields a single exponential, $h^{-2} = ye^{-G}$, while (3.26) gives simply $z = \chi$ (after removing an unimportant infinite constant). In addition, the field strength $F_{(2)}$ is given by (3.18)

$$\widetilde{F}_{(2)} = -2\, d(ye^{-G}) \wedge (dt + V) + 2e^{-G} *_3 d\chi. \tag{3.33}$$

To ensure a solution of the equations of motion, we must also satisfy the second order equations

$$d\Big[*_3 y d(h^2)\Big] = 0, \qquad d\Big[*_3 y^{-1} d\chi\Big] = 0. \tag{3.34}$$

As a result, the solution may be written as

$$\begin{aligned} ds_6^2 &= \mathcal{H}^{-1}(-(dt+V_i dx^i)^2 + d(\phi_2+\chi d\phi_1)^2) + \mathcal{H}(\delta_{ij}dx^i dx^j + dy^2 + y^2 d\phi_1^2), \\ \widetilde{F}_{(2)} &= 2(dt+V)\wedge d(\mathcal{H}^{-1}) - 2\mathcal{H}^{-1}dV, \qquad dV = -\frac{1}{y}*_3 d\chi, \end{aligned} \tag{3.35}$$

where we have introduced the four-dimensional harmonic function $\mathcal{H} = h^2 = \frac{1}{y}e^G$. This generalizes the familiar multi-centered string solution in six-dimensions (which is obtained by taking $\chi = 0$), restricted to singlet configurations along the ϕ_1 direction, assuming that the S^1 parameterized by ϕ_2 has decompactified. Turning on the axion (which also turns on V) yields more general 1/2 BPS solutions of the form obtained in [6].

3.1.3 Only $\widetilde{\eta}$ non-vanishing

With $\eta = 0$ and $\widetilde{\eta} = -1$, the constraint (3.6) indicates that $b = 0$. Setting $a = -\widetilde{\eta} = 1$, we once again see that $e^H = y$. Hence the solutions obtained in this fashion also satisfy (3.29), and thus fall in the same class. In particular, (3.10) and (3.26) gives

$$h^{-2} = ye^G(1+e^{-2G}\chi^2), \qquad z = -\frac{e^{-2G}\chi}{1+e^{-2G}\chi^2} \tag{3.36}$$

(where again an unimportant constant was removed from z).

In fact, these expressions are readily obtained from the previous case of $\eta = 1$, $\widetilde{\eta} = 0$ by performing the $\mathrm{SL}(2,\mathbb{Z})$ transformation $\tau \to -1/\tau$ with the identification

$$h^2 = \frac{1}{y}\Im\tau, \qquad z = \Re\tau. \tag{3.37}$$

In particular, for $\tau = \chi + ie^G$, we see that

$$\tau \to -\frac{1}{\tau} = \frac{-e^{-2G}\chi + ie^{-G}}{1+e^{-2G}\chi^2}. \tag{3.38}$$

Note, also, that the transformation

$$\tau \to \tau + 1 \to -\frac{1}{\tau+1} = \frac{-e^{-2G}(1+\chi)+ie^{-G}}{1+e^{-2G}(1+\chi)^2} \tag{3.39}$$

relates the $\eta = 1$, $\widetilde{\eta} = 0$ solution to the (two charge) $\eta = 1$, $\widetilde{\eta} = -1$ case. In other words, the two U(1) charges naturally form a two-component $\mathrm{SL}(2,\mathbb{Z})$ charge vector $(\eta, \widetilde{\eta})$, and all three examples $(b, a) = (\eta, -\widetilde{\eta}) = (1, 1)$, $(1, 0)$ and $(0, 1)$ fall into the same $\mathrm{SL}(2,\mathbb{Z})$ conjugacy class.

3.1.4 Both η and $\widetilde{\eta}$ vanishing

Finally, the case $\eta = \widetilde{\eta} = 0$ is distinct from the previous ones, as it corresponds to a standard Kaluza-Klein reduction with uncharged Killing spinors. In this case, the constraint (3.6)

becomes trivial, so that a and b may take on arbitrary values. While $(\eta, -\widetilde{\eta}) = (0,0)$ is a $\mathrm{SL}(2,\mathbb{Z})$ singlet, we assume that at least one of a or b is non-vanishing, so that (b,a) remains a $\mathrm{SL}(2,\mathbb{Z})$ doublet. In this case, (3.11) implies that H is a constant, which we take to be zero.

Up to a $\mathrm{SL}(2,\mathbb{Z})$ transformation, we take the simplest case $(b,a) = (1,0)$. For this case, and with $H = 0$, (3.10) and (3.26) gives

$$h^{-2} = e^{-G}, \qquad z = \chi, \tag{3.40}$$

and (3.18) yields

$$\widetilde{F}_{(2)} = -2d(e^{-G}) \wedge (dt + V) - 2e^{-G} dV, \tag{3.41}$$

with $dV = - *_3 d\chi$. In this case, the solution has the form

$$\begin{aligned} ds_6^2 &= \mathcal{H}^{-1}(-(dt + V_i dx^i)^2 + d(\phi_2 + \chi d\phi_1)^2) + \mathcal{H}(\delta_{ij} dx^i dx^j + dy^2 + d\phi_1^2), \\ \widetilde{F}_{(2)} &= 2(dt + V) \wedge d(\mathcal{H}^{-1}) - 2\mathcal{H}^{-1} dV, \qquad dV = - *_3 d\chi, \end{aligned} \tag{3.42}$$

where $\mathcal{H} = h^2 = e^G$. Note that here the equations of motion are

$$d *_3 d\mathcal{H} = 0, \qquad d *_3 d\chi = 0, \tag{3.43}$$

so that both $\mathcal{H}$ and χ are harmonic in $\mathbb{R}^3$ spanned by (x^1, x^2, y). This solution is in fact of the same form as (3.35), and, in the limit of vanishing axion, represents a multi-centered string solution smeared out along the ϕ_1 direction. Note that here both circles have decompactified.

4 Discussion

We have constructed a family of 1/2 BPS solutions of minimal six-dimensional supergravity. These solutions inherit the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ isometries of the T^2 reduction ansatz. The complex structure is parameterized by $\tau = \chi + ie^G$, whereas the volume of T^2 is given by e^H. We have thus generalized our previous $S^1 \times S^1$ reduction ansatz, with the radii of the two circles given by e^{H+G} and e^{H-G}, by allowing for a non-vanishing axion. The $S^1 \times S^1$ solutions were written in terms of a harmonic function on an auxiliary six dimensional space $\mathbb{R}^2 \times \mathbb{R}^4$, just as it was the case for the $S^3 \times S^3$ reduction of type IIB supergravity. However, the $S^1 \times S^1$ reduction turned out to be inconsistent, due to an additional constraint that the four dimensional gauge field had to satisfy: $F_{(2)} \wedge F_{(2)} = 0$. Moreover, this additional constraint translated into another non-linear differential equation which the harmonic function had to obey. This ultimately prohibited the existence of a family of solutions, even though few isolated solutions were found, such as $\mathrm{AdS}_3 \times S^3$, the maximally symmetric plane wave,

and the multi-center string, provided that the six dimensional Killing spinors were carrying some momentum on the two S^1.

The effect of adding the axion among the Kaluza-Klein states to be kept in the reduction is to remove the constraint rendering the bosonic reduction consistent. At the same time, the 1/2 BPS six dimensional solutions are characterized by *two* functions, one being, as before, harmonic on the auxiliary six-dimensional space $\mathbb{R}^2 \times \mathbb{R}^4$, while the other being harmonic on a four dimensional auxiliary space $\mathbb{R}^2 \times \mathbb{R}^2$. This brings a distinct flavor to the 1/2 BPS solutions of minimal six dimensional supergravity (with two U(1) isometries) in comparison to the 1/2 BPS solutions of type IIB supergravity (with SO(4)×SO(4) isometry).

We have also explicitly constructed the Killing spinors associated with the six-dimensional solutions. The Killing spinors are again charged under the two U(1) isometries, but this time their U(1) charges combine into an $\mathrm{SL}(2,\mathbb{Z})$ doublet, and their corresponding solutions are mapped into each other under the action of $\mathrm{SL}(2,\mathbb{Z})$.

In fact, the solutions we have found appear to be a particular case of a larger class of six dimensional D1-D5 solutions with angular momentum obtained by Lunin, Maldacena and Maoz [9] (which in turn are desingularized versions of those constructed in [10]). This is most transparent if we choose to compare one of our solutions (3.35), corresponding to the U(1) charges $\widetilde{\eta} = 0$ and $\eta \neq 0$, to the solution (2.1) in [9]. A brief inspection of (2.1) in [9] reveals that for solutions of minimal six dimensional supergravity we should identify the functions f_1 and f_5, meaning we must enforce $\dot{\vec{F}}(v)\dot{\vec{F}}(v) = 1$. The dictionary between our (3.35) and (2.1) in [9] includes $\mathcal{H} \to f_1, V \to A_i dx^i, \chi d\phi_1 \to B_i dx^i, \{x^1, x^2, y, \phi_1\} \to \{\vec{x}\}$ and $\phi_2 \to y$. Our solutions have also an additional Killing vector, namely ∂_{ϕ_1}. This restricts the profile $\vec{F}(\vec{x}(v))$ dependence to $\vec{F}(x^1(v), x^2(v))$ at $y = 0$. The solution (2.1) in [9], which was derived by applying a chain of dualities to a fundamental string carrying momentum, was shown to be regular provided that the profile of the fundamental string, specified by $\vec{F}(v)$, obeyed a few conditions: it was not self-intersecting, and $|\dot{\vec{F}}(v)| \neq 0$.

The main outcome of this comparison between our solution (3.35) and (2.1) of [9] is that we realize that subsequent regularity conditions will relate the boundary conditions of our two harmonic functions: χ/y^2 and $h^2 = \mathcal{H}$, since equation (2.2) of [9] can be rewritten in terms of Green's function associated with our harmonic functions. Therefore the bubbling picture for $\mathrm{AdS}_3 \times S^3$ is completed upon enforcing regularity, and the two dimensional droplets of the bubbling $\mathrm{AdS}_5 \times S^5$ solution have morphed into boundaries specified by the profile $\vec{F}(v)$. Understanding the regularity properties of our solutions and their direct relationship with the chiral primaries of the dual CFT deserves further study. It would also be desirable to understand the peculiarities of the giant gravitons (their unrestricted growth, their discrete angular momenta) in terms of the bubbling AdS_3 picture.

Note added

While this work was under completion, we became aware of [8], where bubbling $\mathrm{AdS}_3 \times S^3$ solutions of the form (3.31) were also obtained. The analysis of [8] followed directly from the complete six-dimensional classification of [6] by choosing an appropriate reduction with three commuting Killing symmetries $\partial/\partial x^+$, $\partial/\partial x^-$ and $\partial/\partial\phi$. From a six-dimensional point of view, the Killing vector $K^M = \bar\epsilon\Gamma^M\epsilon$ is null, leading to a natural x^+, x^- basis. To make the comparison more direct, we may invert the expressions (3.28) and (3.30) to obtain

$$e^{-G} = h^2y + (h^2y)^{-1}(z-\tfrac{1}{2})^2, \qquad \chi = -\frac{h^2y + (h^2y)^{-1}(z^2-\frac{1}{4})}{h^2y + (h^2y)^{-1}(z-\frac{1}{2})^2}, \tag{4.1}$$

which correspond to the metric elements given in [8].

Furthermore, the work of [8] demonstrates that the bubbling $\mathrm{AdS}_3 \times S^3$ solutions are in fact a restricted sub-class of *all* the 1/2 BPS solutions of [6]. This brings up an appropriate note of caution, namely that the bubbling forms of 1/2 BPS solutions are not necessarily exhaustive, as far as the full theory is concerned, but only correspond to sub-classes where additional Killing symmetries are imposed on the background.

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A Integrability of the Killing spinor equations

In [1], the integrability of the supersymmetry variations (2.16) was obtained in the absence of the axion. The results of that work is easily extended to the present case. We take

$$\delta\psi_\mu = \mathcal{D}_\mu\epsilon, \qquad \delta\lambda_H = \Delta_H, \qquad \delta\lambda_G = \Delta_G, \tag{A.1}$$

where, from (2.16), we read off

$$\begin{aligned}
\mathcal{D}_\mu &= \nabla_\mu - \tfrac{i}{4}\gamma^5 e^{-G}\partial_\mu\chi + \tfrac{i}{16}e^{-\frac{1}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu,\\
\Delta_H &= \gamma^\mu\partial_\mu H + e^{-\frac{1}{2}H}((\eta-\chi\widetilde{\eta})e^{-\frac{1}{2}G} - i\widetilde{\eta}\gamma_5 e^{\frac{1}{2}G}),\\
\Delta_G &= \gamma^\mu\partial_\mu G + i\gamma^5\gamma^\mu e^{-G}\partial_\mu\chi - \tfrac{i}{4}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu} + e^{-\frac{1}{2}H}((\eta-\chi\widetilde{\eta})e^{-\frac{1}{2}G} + i\widetilde{\eta}\gamma_5 e^{\frac{1}{2}G}).
\end{aligned} \tag{A.2}$$

Here we may read off three independent integrability conditions, related to the commutators $[\mathcal{D}_\mu, \mathcal{D}_\nu]$, $[\mathcal{D}_\mu, \Delta_H]$ and $[\mathcal{D}_\mu, \Delta_G]$. For $[\mathcal{D}_\mu, \mathcal{D}_\nu]$, we obtain

$$\begin{aligned}
\gamma^\mu[\mathcal{D}_\mu, \mathcal{D}_\nu] \;=\; & \tfrac{1}{2}[R_{\nu\sigma} - \tfrac{1}{4}e^{-(H-G)}(\widetilde{F}^2{}_{\nu\sigma} - \tfrac{1}{4}g_{\mu\nu}\widetilde{F}^2) \\
& \quad -\tfrac{1}{2}(\partial_\nu H\partial_\sigma H + \partial_\nu G\partial_\sigma G + e^{-2G}\partial_\nu\chi\partial_\sigma\chi) - \nabla_\nu\nabla_\sigma H]\gamma^\sigma \\
& + \tfrac{i}{16}e^{-\frac{1}{2}(H+G)}(\partial_{[\mu}F_{\lambda\sigma]} + \widetilde{F}_{[\mu\lambda}\partial_{\sigma]}\chi)\gamma^{\mu\lambda\sigma}\gamma_\nu + \tfrac{i}{8}e^{-\frac{1}{2}(H-G)}\nabla^\mu(e^{-G}F_{\mu\lambda})\gamma^\lambda\gamma_\nu \\
& + \tfrac{1}{2}[\mathcal{D}_\nu, \Delta_H] + \tfrac{1}{8}\partial_\nu(H+G)(\Delta_H + \Delta_G) \\
& + \tfrac{1}{8}[\partial_\nu(H-G) - \tfrac{i}{4}e^{-\frac{1}{2}(H+G)}F_{\lambda\sigma}\gamma^{\lambda\sigma}\gamma_\nu + 2i\gamma^5 e^{-G}\partial_\nu\chi](\Delta_H - \Delta_G). \qquad \text{(A.3)}
\end{aligned}$$

Note that we have used the relation $F_{(2)} = e^G *_4 \widetilde{F}_{(2)}$ to rewrite the Einstein equation in terms of $\widetilde{F}_{(2)}$. Since the last two lines above vanish on Killing spinors, this integrability condition yields the Einstein equation in conjunction with the Bianchi identity and equation of motion for $\widetilde{F}_{(2)}$.

Turning to the $[\mathcal{D}_\mu, \Delta_H]$ condition, we find

$$\begin{aligned}
\gamma^\mu[\mathcal{D}_\mu, \Delta_H] \;=\; & \Box H + \partial H^2 \\
& -[\gamma^\mu\partial_\mu H - \tfrac{i}{2}\gamma^5\gamma^\mu e^{-G}\partial_\mu\chi + \tfrac{i}{8}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu} \\
& \qquad -\tfrac{1}{2}e^{-\frac{1}{2}H}((\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}G} + i\widetilde{\eta}\gamma_5 e^{\frac{1}{2}G})]\Delta_H \\
& -\tfrac{1}{2}e^{-\frac{1}{2}H}((\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}G} - i\widetilde{\eta}\gamma_5 e^{\frac{1}{2}G})\Delta_G. \qquad \text{(A.4)}
\end{aligned}$$

On Killing spinors this yields precisely the H equation of motion, $\nabla^\mu\left(e^H\nabla_\mu H\right) = 0$, of (2.12). This indicates that the H equation of motion (and hence the solution for H) is guaranteed by supersymmetry.

Finally, the $[\mathcal{D}_\mu, \Delta_G]$ integrability condition becomes

$$\begin{aligned}
\gamma^\mu[\mathcal{D}_\mu, \Delta_G] \;=\; & \Box G + \partial H\partial G + e^{-2G}(\partial\chi)^2 - \tfrac{1}{8}e^{-(H-G)}\widetilde{F}^2 \\
& -i\gamma^5 e^{-G}[\Box\chi + \partial(H-2G)\partial\chi + \tfrac{1}{16}e^{-H+2G}\epsilon_{\mu\nu\lambda\sigma}\widetilde{F}_{\mu\nu}\widetilde{F}_{\lambda\sigma}] \\
& -\tfrac{i}{4}e^{-\frac{1}{2}(H+G)}(\partial_{[\mu}F_{\nu\lambda]} + \widetilde{F}_{[\mu\nu}\partial_{\lambda]}\chi)\gamma^{\mu\nu\lambda} - \tfrac{i}{2}e^{-\frac{1}{2}(H-G)}\nabla^\mu(e^{-G}F_{\mu\nu})\gamma^\nu \\
& -\tfrac{1}{2}[\gamma^\mu\partial_\mu G - i\gamma^5\gamma^\mu e^{-G}\partial_\mu\chi]\Delta_H - \tfrac{1}{2}[\gamma^\mu\partial_\mu H + i\gamma^5\gamma^\mu e^{-G}\partial_\mu\chi]\Delta_G. \quad \text{(A.5)}
\end{aligned}$$

In addition to the Bianchi identity and equation of motion for $\widetilde{F}_{(2)}$, this condition yields the equations of motion for the $SL(2,\mathbb{R})$ scalar $\tau = \chi + ie^G$. In the absence of an axion, it is this equation that leads to the $\widetilde{F}_{(2)} \wedge \widetilde{F}_{(2)} = 0$ constraint of [1].

Disregarding the H equation, which is automatically satisfied on a supersymmetric background, we see that the existence of a Killing spinor only ensures that linear combinations of the Einstein equation, $\widetilde{F}_{(2)}$ Bianchi identity and equation of motion, and τ equation of motion are satisfied. Although somewhat more care is needed to fully disentangle the bosonic equations of motion in (A.3) and (A.5), we see that, so long as the $\widetilde{F}_{(2)}$ Bianchi identity and

equation of motion are satisfied, (A.3) then guarantees that the Einstein equation will hold, and further (A.5) will ensure the full τ equation of motion (so long as the Killing spinor has indefinite γ^5 chirality). We thus conclude that, for obtaining supersymmetric backgrounds, it would be sufficient to satisfy the $\widetilde{F}_{(2)}$ Bianchi identity and equation of motion in addition to the Killing spinor equations themselves.

B Differential identities for the spinor bilinears

The supersymmetric construction of [4–7] proceeds by postulating the existence of a Killing spinor ϵ and then forming the tensors f_1, f_2, K_μ, L_μ and $Y_{\mu\nu}$ from spinor bilinears (3.1). The algebraic identities of interest were given in the text in (3.2). Here we tabulate the differential identities obtained by demanding that ϵ solves the Killing spinor equations obtained from (2.16).

First, by assuming $\delta\psi_\mu = 0$, we may demonstrate that

$$\begin{aligned}
\nabla_\mu f_1 &= \tfrac{1}{4}e^{-\frac{1}{2}(H+G)} * F_{\mu\nu}K^\nu + \tfrac{1}{2}f_2 e^{-G}\partial_\mu\chi,\\
\nabla_\mu f_2 &= -\tfrac{1}{4}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}K^\nu - \tfrac{1}{2}f_1 e^{-G}\partial_\mu\chi,\\
\nabla_\mu K_\nu &= \tfrac{1}{4}e^{-\frac{1}{2}(H+G)}(f_2F_{\mu\nu} - f_1 * F_{\mu\nu}),\\
\nabla_\mu L_\nu &= \tfrac{1}{4}e^{-\frac{1}{2}(H+G)}(\tfrac{1}{2}g_{\mu\nu}F_{\lambda\rho}Y^{\lambda\rho} - 2F_{(\mu}{}^\lambda Y_{\nu)\lambda}),\\
\nabla_\mu Y_{\nu\lambda} &= \tfrac{1}{4}e^{-\frac{1}{2}(H+G)}(2g_{\mu[\nu}F_{\lambda]\rho}L^\rho - 2F_{\mu[\nu}L_{\lambda]} + F_{\nu\lambda}L_\mu).
\end{aligned}\tag{B.1}$$

In particular, the equation for K_μ indicates that $K_{(\mu;\nu)} = 0$, so that K^μ is Killing. This is in fact a generic feature of constructing a Killing vector from Killing spinors.

In addition, the $\delta\chi_H = 0$ condition allows us to derive the additional relations

$$\begin{aligned}
&K^\mu\partial_\mu H = 0, \qquad (\eta - \chi\widetilde{\eta})f_2 = -\widetilde{\eta}e^G f_1,\\
&L^\mu\partial_\mu H = (\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}(H+G)}f_1 - \widetilde{\eta}e^{-\frac{1}{2}(H-G)}f_2,\\
&(\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}(H+G)}L_\mu = f_1\partial_\mu H, \qquad \widetilde{\eta}e^{-\frac{1}{2}(H-G)}L_\mu = -f_2\partial_\mu H,\\
&(\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}(H+G)}K_\mu = *Y_\mu{}^\nu\partial_\nu H, \qquad \widetilde{\eta}e^{-\frac{1}{2}(H-G)}K_\mu = Y_\mu{}^\nu\partial_\nu H,\\
&2L_{[\mu}\partial_{\nu]}H = 0, \qquad 2K_{[\mu}\partial_{\nu]}H = (\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}(H+G)} * Y_{\mu\nu} + \widetilde{\eta}e^{-\frac{1}{2}(H-G)}Y_{\mu\nu}.
\end{aligned}\tag{B.2}$$

Similarly, the $\delta\chi_G = 0$ condition yields the relations

$$\begin{aligned}
&K^\mu\partial_\mu G = 0, \qquad \tfrac{1}{4}F_{\mu\nu} * Y^{\mu\nu} = (\eta - \chi\widetilde{\eta})f_2 - \widetilde{\eta}e^G f_1,\\
&L^\mu\partial_\mu G = (\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}(H+G)}f_1 + \widetilde{\eta}e^{-\frac{1}{2}(H-G)}f_2 - \tfrac{1}{4}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}Y^{\mu\nu},\\
&(\eta - \chi\widetilde{\eta})e^{-\frac{1}{2}(H+G)}L_\mu = f_1\partial_\mu G + \tfrac{1}{2}e^{-\frac{1}{2}(H+G)} * F_{\mu\nu}K^\nu,\\
&\qquad \widetilde{\eta}e^{-\frac{1}{2}(H-G)}L_\mu = f_2\partial_\mu G + \tfrac{1}{2}e^{-\frac{n}{2}(H+G)}F_{\mu\nu}K^\nu,
\end{aligned}$$

$$
\begin{aligned}
(\eta-\chi\widetilde{\eta}e^{-\frac{1}{2}(H+G)}K_\mu &= *Y_\mu{}^\nu\partial_\nu G+\tfrac{1}{2}e^{-\frac{1}{2}(H+G)}*F_{\mu\nu}L^\nu,\\
\widetilde{\eta}e^{-\frac{1}{2}(H-G)}K_\mu &= -Y_\mu{}^\nu\partial_\nu G+\tfrac{1}{2}e^{-\frac{1}{2}(H+G)}F_{\mu\nu}L^\nu,\\
2L_{[\mu}\partial_{\nu]}G &= 2e^{-\frac{1}{2}(H+G)}F_{[\mu}{}^\rho Y_{\nu]\rho},\\
2K_{[\mu}\partial_{\nu]}G &= (\eta-\chi\widetilde{\eta})e^{-\frac{1}{2}(H+G)}*Y_{\mu\nu}-\widetilde{\eta}e^{-\frac{1}{2}(H-G)}Y_{\mu\nu}-\tfrac{1}{2}e^{-\frac{1}{2}(H+G)}(f_1*F_{\mu\nu}+f_2F_{\mu\nu}).
\end{aligned}
\tag{B.3}
$$

Although the above identities are algebraic and not differential on the spinor bilinears, they originate from the supersymmetry variations along the internal directions of the Kaluza-Klein reduction. So in this sense, they form a generalized set of 'differential identities'. However, as they are only algebraic, they prove extremely useful in determining much of the geometry, as is evident from the analysis of [2].

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